Low-Frequency Small-Signal Models (using unilateral model for Early Effect)

We will now motivate the use of the following small signal models.

<table>
<thead>
<tr>
<th></th>
<th>$r_\pi$</th>
<th>$g_m$</th>
<th>$g_s$</th>
<th>$r_0$</th>
<th>$g_m r_0$</th>
<th>$g_m r_\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sub-VT mos (sat)</td>
<td>$\infty$</td>
<td>$\frac{\kappa I}{\phi_t} = \kappa g_s$</td>
<td>$\frac{l}{\phi_t}$</td>
<td>$\frac{V_A}{I}$</td>
<td>$\frac{V_A}{\phi_t}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>above-VT mos ((\kappa \approx 1))(sat)</td>
<td>$\infty$</td>
<td>(\sqrt{2lk'} = \frac{2l}{V_g - V_s - V_T} = \frac{2l}{V_{on}})</td>
<td>$g_m$</td>
<td>$\frac{V_A}{I}$</td>
<td>$\frac{2V_A}{V_g - V_s - V_T} = \frac{2V_A}{V_{on}}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>BJT (fwd. active)</td>
<td>$\frac{\beta \phi_t}{I_C}$</td>
<td>$\frac{l_c}{\phi_t}$</td>
<td>$g_m$</td>
<td>$\frac{V_A}{I_C}$</td>
<td>$\frac{V_A}{\phi_t}$</td>
<td>$\beta$</td>
</tr>
</tbody>
</table>

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1st-order system with one capacitor to ground. Therefore, the state variable $V_o$ is easily derived from $I_c$:

$$\frac{dV_{\text{out}}}{dt} = \frac{I_c}{C} = \frac{l_1 + l_2 - l_3}{C} = \frac{V_i - V_o}{R_1} + \frac{V_2 - V_o}{R_2} - I_3$$

DC condition (with attention to input): $\frac{dV_{\text{out}}}{dt} = 0$

$$\frac{V_i - V_o}{R_1} + \frac{V_2 - V_o}{R_2} - l_3 = 0$$

$$V_o \frac{1}{R_1 || R_2} = V_i \frac{1}{R_1} + V_2 \frac{1}{R_2} - l_3$$

$$V_o = (V_i \frac{1}{R_1} + V_2 \frac{1}{R_2} - l_3) (R_1 || R_2)$$

$$V_o = V_i \frac{1}{R_1} (R_1 || R_2) + \left( V_2 \frac{1}{R_2} - l_3 \right) (R_1 || R_2)$$

DC transfer function

$$\text{constant } V_{o,\text{offset}}$$
DC Gain from DC Transfer Function

DC transfer function:

\[ V_o = V_i \frac{1}{R_1} (R_1 || R_2) + \left( V_2 \frac{1}{R_2} - I_3 \right) (R_1 || R_2) \]

DC gain:

\[ A_{DC} = \frac{dV_o}{dV_i} \]

\[ A_{DC} = \frac{1}{R_1} (R_1 || R_2) \]

In a linear circuit, DC gain does not depend on any independent sources (current or voltage). Also note the lack of C in this answer.
DC Gain is Not

\[ A_{DC} = \frac{dV_o}{dV_i} = \frac{1}{R_1} (R_1 || R_2) \]

\[ A_{DC} \neq \frac{V_o}{V_i} = \frac{1}{R_1} (R_1 || R_2) + \frac{\left( V_2 \frac{1}{R_2} - I_3 \right) (R_1 || R_2)}{V_i} \]
The following data is observed. What is the gain? What assumptions are you making in your answer?

<table>
<thead>
<tr>
<th>$V_i$</th>
<th>$V_o$</th>
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<td>1mV</td>
<td>1.75V</td>
</tr>
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The following data is observed. What is the gain? What assumptions are you making in your answer?

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<td>1mV</td>
<td>1.75V</td>
</tr>
<tr>
<td>2mV</td>
<td>2.5V</td>
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</table>
Ex(3):

What is the missing value? What are your assumptions?

<table>
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<th>$V_i$</th>
<th>$V_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1mV</td>
<td>2V</td>
</tr>
<tr>
<td>2mV</td>
<td>2.5V</td>
</tr>
<tr>
<td>0</td>
<td>?</td>
</tr>
</tbody>
</table>
One line assumes a linear proportionality between $V_i$ and $V_o$, the other only assumes a linear proportionality between changes in $V_i$ and $V_o$. 
Superposition (I)

Original Linear Circuit

Superposition Circuit

DC Operating Point Circuit

Perturbation Circuit

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Lecture III
1st-order system from before with one capacitor to ground. Therefore, the state variable $V_o$ is easily derived from $I_c$:

\[
\frac{dV_o}{dt} = \frac{I_c}{C} = \frac{l_1+l_2-l_3}{C} \quad \text{current input}
\]

\[
\frac{dV_o}{dt} = \frac{V_1-V_0}{R_1} + \frac{V_2-V_0}{R_2} - l_3 \quad \text{voltage input}
\]

Two choices with same result

- Separating Current e.g.: $I_1 = I_1 + \Delta l_1$
- Separating Voltage e.g.: $V_1 = V_1 + \Delta V_1$

Separating Voltage (Superposition Circuit):

\[
\frac{d(V_{oDC}+\Delta V_o)}{dt} = \frac{dV_{oDC}}{dt} + \frac{d\Delta V_o}{dt} = \left(-\left(V_{oDC}+\Delta V_o\right) \frac{1}{(R_1||R_2)} + \frac{(V_{iDC}+\Delta V_i)}{R_1} + \frac{(V_{2DC}+\Delta V_2)}{R_2} - (l_{3DC} + \Delta l_3) \right) \frac{1}{C}
\]
define \( \frac{dV_{oDC}}{dt} = 0 \)

Operating Point (when \( \Delta V_o = 0, \Delta V_i = 0 \)):

\[
\frac{dV_{oDC}}{dt} + \frac{d\Delta v_o}{dt} = \left( -\left( V_{oDC} \right) \frac{1}{(R_1||R_2)} + \left( V_{iDC} \right) \frac{1}{R_1} + \left( V_{2DC} \right) \frac{1}{R_2} - \left( I_{3DC} \right) \right) \frac{1}{C}
\]

\[
0 = \left( -\left( V_{oDC} \right) \frac{1}{(R_1||R_2)} + \left( V_{iDC} \right) \frac{1}{R_1} + \left( V_{2DC} \right) \frac{1}{R_2} - \left( I_{3DC} \right) \right) \frac{1}{C}
\]

DC Condition from before

(Small-Signal) Perturbation model

\[
\frac{d\Delta v_o}{dt} = \left( -\left( \Delta V_o \right) \frac{1}{(R_1||R_2)} + \left( \Delta V_i \right) \frac{1}{R_1} + \left( \Delta V_2 \right) \frac{1}{R_2} - \left( \Delta I_3 \right) \right) \frac{1}{C} + \left( -\left( V_{oDC} \right) \frac{1}{(R_1||R_2)} + \left( V_{iDC} \right) \frac{1}{R_1} + \left( V_{2DC} \right) \frac{1}{R_2} - \left( I_{3DC} \right) \right) \frac{1}{C}
\]

0 according to operating point result
(Small-Signal) Perturbation model continued

\[ \frac{d\Delta v_o}{dt} = \left( - (\Delta V_o) \right) \frac{1}{(R_1||R_2)} + \frac{(\Delta V_i)}{R_1} + \frac{\left( \frac{\Delta V_2}{R_2} \right)^0}{R_2} - \left( \frac{\Delta I_3}{R_1} \right)^0 \frac{1}{C} \]

“Simplified” (Small-Signal) Perturbation Model:

\[ \frac{d\Delta v_o}{dt} + \Delta V_o \frac{1}{(R_1||R_2) C} = \left( \frac{\Delta V_i}{R_1 C} \right) \]
\[ \frac{d \Delta v_o}{dt} + \Delta v_o \left( \frac{1}{R_1 \| R_2} \right) C = \left( \frac{\Delta v_i}{R_1 C} \right) \]

(Small-Signal) Perturbation DC gain:
Let \( \frac{d \Delta v_o}{dt} = 0 \):

\[ 0 + \Delta v_o \left( \frac{1}{R_1 \| R_2} \right) C = \left( \frac{\Delta v_i}{R_1 C} \right) \]

\[ \frac{\Delta v_o}{\Delta v_i} = \left( \frac{1}{R_1} \right) \left( R_1 \| R_2 \right) \]

same as before: Can calculate DC gain from full model, or (small-signal) perturbation model
Using well-known analysis of RC circuits:

**Small-Signal Step response**

\[ \Delta v_i = \alpha u(t) \]
\[ \Delta v_o = \alpha A \left( 1 - e^{-t/\tau} \right) \text{ where } \tau = \frac{1}{(R_1 || R_2)C} ; A = \left( \frac{1}{R_1} \right) (R_1 || R_2) \]

**Small-Signal Sinusoidal Response**

\[ \Delta v_i = \alpha \sin(\omega t - \phi) \]
\[ \Delta v_o = \alpha A \frac{1}{\sqrt{1 + (\omega r_{eq} C)^2}} \cos \left( \omega t - \phi - \tan^{-1}(\omega r_{eq} C) \right) \]

where \( A = \frac{1}{R_1} r_{eq} \) and \( r_{eq} = R_1 || R_2 \)
Simplifying Assumption

For this first example, let's work with $\lambda=0$ to make the math simple for the moment by removing the $I_{ds}$ dependence on $V_o$.

\[
\frac{dV_o}{dt} = \left( \frac{V_{dd} - V_o}{R} \right) - \frac{k'}{2} \frac{W}{L} (V_{ov})^2
\]
assuming saturation and $\lambda=0$

\[
\frac{dV_o}{dt} = \left( \frac{V_{dd} - V_o}{R} \right) - \frac{k'}{2} \frac{W}{L} (V_{ov})^2
\]
where $V_{ov} = V_g - V_s - V_T$
Non-linear circuit \((2) (\lambda=0)\)

DC condition:

\[
\left( \frac{V_{dd} - V_o}{R} \right) = \frac{k'}{2} \frac{W}{L} \left( V_g - V_s - V_T \right) \left( V_{ov} \right)^2
\]

\[V_o = V_{dd} - \frac{k'}{2} \frac{W}{L} V_{ov}^2 R\]

DC gain from DC condition:

\[
\frac{dV_o}{dV_i} = -k' V_{ov} \frac{W}{L} R
\]

We see that the gain now depends on the operating point.

Here is another form pointing out dependence on bias current:

\[
\frac{dV_o}{dV_i} = -2 \left( \frac{k'}{2} \frac{V_{ov}^2 W}{L} V_{ov} \right) R = -\frac{2I}{V_{ov}} R
\]
studying the perturbation directly is more difficult than before

\[
\frac{d\Delta V_o}{dt} = \left( \frac{V_{dd} - (V_o + \Delta V_o)}{R} \right) - \frac{k'}{2} \frac{W}{L} (V_{ov} + \Delta V_{ov})^2 \frac{C}{C}
\]

\[
\left( \frac{V_{dd} - V_o}{R} \right) - \frac{k'}{2} \frac{W}{L} \left( V_{ov}^2 + 2V_{ov} \Delta V_{ov} + \Delta V_{ov}^2 \right)
\]

\[
\frac{d\Delta V_o}{dt} - \Delta V_o \frac{1}{RC} = \frac{C}{C}
\]

And other various analysis including perturbation analysis is similarly difficult and often best left to a simulator

- We can replace a non-linear circuit with a linear model valid in the vicinity an operating point. The parameters of the linear model are based on the operating point.
- Can also “easily” reinstate $I_{ds}$ dependence $V_o (\lambda \neq 0)$
Common source amplifier

Original Non-linear Circuit

Linearized Circuit

DC Operating Point Circuit

Perturbation Circuit

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Common source amplifier

Original Non-linear Circuit

Determine Operating Model
And DC Operating Point

Select Appropriate
Transistor Model Based on
Transistor Operating Region/Mode

Linearized Small-Signal Perturbation Circuit

Determine Numerical Values of Small-Signal Parameters (gm, gd, rd, etc...)

Numerical Results

Determine Analytical Model of DC Gain, Frequency Dependency, Rout, Gm, etc....
with symbolic small-signal parameters gm, gd, rd, etc...
Small-Signal Analysis (3)

Original Non-linear Circuit

\[ \frac{dV_o}{dt} = -v_i g_m - \frac{v_o}{(R_D || r_d)} \]

Steady State Condition: set \( \frac{dV_o}{dt} = 0 \)

\(-v_i g_m = \frac{v_o}{(R_D || r_d)} \)

Small Signal DC gain:

\[ A = \frac{v_o}{v_i} = -g_m (R_D || r_d) \]

\( A = \frac{v_o}{v_i} \)

(OK in small-signal world. Doesn’t contradict what we said before)
A Divide and Conquer Approach that works quite often with amplifier circuits:

1. With the output voltage held fixed, calculate the output current perturbation based on the input and parameterize with value:
   \[ G_m = \frac{dI_{out}}{dV_{in}} \]

2. With no input, calculate the effect of an current perturbation on the output voltage (how far it must move in response to counteract a current perturbation) and parameterize with value \( R_{out} = \frac{dV_{out}}{dI_{out}} \).

Finally: \( A_{DC} = G_m R_{out} \)
Small-Signal Analysis (5) Finding $G_m$

- Apply test voltage $v_t$.
- Modulate input around operating point.
- Hold output fixed to no modulation.
- Measure output current (zero when $V_i = V_{iDC}$).

\[ G_m = \frac{i_y}{v_t} \]

\[ i_y = -v_t g_m \]

\[ \frac{i_y}{v_t} = -g_m \]

\[ G_m = -g_m \]
Small-Signal Analysis (6) Finding Rout with test current

- Set input to zero
- Set input to DC operating point
- Apply small current to output and measure effect on output voltage

$$ R_{out} = \frac{v_y}{i_t} $$

$$ i_t = \frac{v_y}{(R_D || r_d)} $$

$$ R_{out} = (R_D || r_d) $$
Small-Signal Analysis (7) Finding Rout with test voltage

\[ R_{out} = \frac{v_t}{i_y} \]

\[ i_y = \frac{v_t}{(R_D||r_d)} \]

\[ \frac{v_t}{i_y} = (R_D||r_d) \]

\[ R_{out} = (R_D||r_d) \]
Small-Signal Analysis (8) \( A_v = G_m \times R_{out} \)

\[
G_m = -g_m \\
R_{out} = (R_D || r_d) \\
A_v = G_m \times R_{out} = -g_m (R_D || r_d)
\]

Furthermore, the output time constant is

\[
\tau = R_{out} C_{out} = (R_D || r_d) C_{out}
\]